WAVES AT THE FLUID SURFACE DURING THE MOTION OF A SUBMERGED ELLIPSOID OF REVOLUTION

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A problem of determining the form assumed by the free surface of a perfect fluid of infinite depth during the motion of a submerged ellipsoid of revolution is considered.

A problem of this kind (for the case of a moving pressure impulse) was first solved by Kelvin [1]. However in view of the computational difficulties arising, the asymptotic values for the wave ordinates at large distances from the source of perturbations [2 - 5] were the only ones obtained up to the present time.

The method presented below makes possible the computation of the exact (in the linear formulation) values of the ordinates of the free surface during the motion of an ellipsoid of revolution with a large aspect ratio, with very moderate consumption of the computer time. The results obtained may find application in the theoretical problems of ship-building, hydrotechnology, etc.

1. Assuming the liquid perfect and the waves appearing at the free surface small, we shall solve the problem using the method of singularities. Then the velocity potential resulting from the linear steady motion of the ellipsoid moving at velocity v can be written in the form [6]

$$\Phi(x, y, z) = \int_S \varphi_0(x, y, z, \xi, \eta, \zeta)\, q(\xi, \eta, \zeta)\, dS \tag{1.1}$$

Here φ_0 is the potential of a unit source satisfying the following boundary conditions at the free surface

$$\frac{\partial \varphi_0}{\partial x^2} + \nu \frac{\partial \varphi_0}{\partial z}\bigg|_{z=0} = 0 \qquad (\nu = g / v^2) \tag{1.2}$$

and the corresponding conditions at infinity, and q denotes the intensity of the equivalent singularities determined from the condition of zero flow across the surface S of the ellipsoid.

Following [6] we assume that the intensity q is equal, in its first approximation, to that occurring during the motion of the ellipsoid in an unbounded fluid. (Of course, the error decreases with increasing both the depth of immersion of the ellipsoid and its aspect ratio). As we know [7], the flow about the ellipsoid of revolution is in this case equivalent to a flow about the sources and sinks distributed continuously between its foci. Indeed, using the cylindrical x, ρ_0, θ coordinates in the expression for the potential of an ellipsoid of revolution moving along the x-axis [1] we can obtain

$$\Phi = \frac{A}{2l}\left[x \ln\left|\frac{r_1 + r_2 + 2l}{r_1 + r_2 - 2l}\right| - r_1 + r_2\right]$$

$$r_1 = [(x+l)^2 + \rho_0{}^2]^{1/2}, \qquad r_2 = [(x-l)^2 + \rho_0{}^2]^{1/2}$$

where l is the distance between the foci and the coordinate origin. This expression represents the result of integrating the potential of elementary sources distributed along the x-axis the intensity of which varies according to the linear law

$$\Phi = \int_{-l}^{l} \varphi(x-\xi)\, q(\xi)\, d\xi \tag{1.3}$$

$$\varphi(x-\xi) = \frac{-1}{4\pi} \sqrt{(x-\xi)^2 + \rho_0{}^2}, \qquad q = 2\pi A \frac{\xi}{l}$$

$$A = va\left[\frac{1}{1-e^2} - \frac{1}{2e}\ln\left|\frac{1+e}{1-e}\right|\right], \qquad e = \frac{l}{a}, \qquad a = \zeta l$$

Thus under the above assumptions our problem can be solved by inserting the potential of the source situated near the free surface and satisfying the boundary condition (1.2), into the right-hand side of (1.3).

2. The corresponding expression for the potential of a unit source situated at the point $(0,\ 0,\ z_1)$ of the moving coordinate system can be obtained by performing a passage to the limit as $t \to \infty$ in the problem with initial conditions [3]. Then we have

$$\varphi = \frac{1}{4\pi}\left\{\frac{1}{r'} - \frac{1}{r} - \frac{\nu}{\pi}\int_0^\infty e^{k\,(z+z_1)} \int_C e^{ik(x\cos\theta + y\sin\theta)}\, \frac{d\theta dk}{\nu - k\cos^2\theta}\right\}$$

$$r = [x^2 + y^2 + (z-z_1)^2]^{-1/2}, \qquad r' = [x^2 + y^2 + (z+z_1)^2]^{-1/2} \tag{2.1}$$

The contour C passes along the real axis from $\theta = -\pi$ to $\theta = \pi$ and, in accordance with the condition that no waves are present at large distances in front of the ellipsoid, it is indented about the singularities $\theta = \pm\sqrt{\arccos\,(\nu / k)}$ from below when $\theta > 0$ and from above when $\theta < 0$

The singularity appearing in the denominator of the inner integral of (2.1) complicates the computations appreciably. It may however be removed by transforming the contour C into a closed one by means of the variable substitution $\beta = e^{i\theta}$. Then. after calculating the corresponding residues, we set $k / \nu = 1 - t^2$ for $k < \nu$ and $k / \nu = 1 + t^2$ for $k > \nu$ to obtain

$$\varphi = \frac{\nu}{\pi}\left\{\frac{1}{4r'} - \frac{1}{4r} - \int_0^1 e^{(1-t^2)(z+z_1)} \cos(x\sqrt{1-t^2})\,\mathrm{ch}\,(yt\sqrt{1-t^2})\, dt + \right.$$

$$+ \int_0^\infty e^{(1+t^2)(z+z_1)} \sin(x\sqrt{1+t^2}) \cos(yt\sqrt{1-t^2})\, dt + \int_0^1 e^{(1-t^2)(z+z_1)} \sum_{m=1}^{\infty} (-1)^m \left[\left(\frac{1+t}{1-t}\right)^m - \right.$$

$$\left. - \left(\frac{1-t}{1+t}\right)^m\right] J_{2m}\,[\rho(1-t^2)] \cos 2m\psi dt + 2\int_0^\infty e^{(1+t^2)(z+z_1)} \sum_{m=1}^{\infty} (-1)^m \sin(2m\,\mathrm{arctg}\, t) \times \tag{2.2}$$

$$\left. \times J_{2m}\,[\rho(1+t^2)] \cos 2m\psi dt\right\}$$

$$\rho = (x^2 + y^2)^{1/2}, \qquad \psi = \mathrm{arctg}\;(y / x)$$

where J_{2m} is the Bessel function of the first kind and all coordinates refer to ν^{-1}.

The above expression for the potential has the following shortcomings. When the values of $|\, z + z_1 \,|$ are small, the improper integrals converge slowly. Also, when ρ are large, a considerable number of summation terms must be computed. For such values of coordinates it is more convenient to perform the integration in (2.1) along the imaginary axis [8] to obtain

$$\varphi = \frac{\nu}{\pi}\left\{\frac{1}{4r'} - \frac{1}{4r} + [1 - \operatorname{sign}(x)] \int_0^\infty e^{(1+t^2)(z+z_1)} \sin (x\sqrt{1+t^2}) \cos (yt\sqrt{1+t^2})\, dt - \right.$$

$$\left. - \frac{1}{\pi}\int_0^\infty e^{-|x|t} dt \int_0^1 \cos(\alpha t y) \frac{\sqrt{1-\alpha^2}\cos[(z+z_1)t\sqrt{1-\alpha^2}] - t\sin[(z+z_1)t\sqrt{1-\alpha^2}]}{1-\alpha^2+t^2}\, d\alpha \right\}$$

3. Let us insert the expressions obtained into (1.3). The following expression can be obtained for the ordinates ζ_w of the free surface:

$$\zeta_w = \frac{\zeta_w g}{v^2} = \frac{1}{v}\frac{\partial \Phi}{\partial x} = \frac{2\pi A}{vl}[\varphi^*(x+l) - \varphi^*(x-l) - l\varphi(x+l) - l\varphi(x-l)] \qquad (3.1)$$

$$\varphi^*(x) = \int_0^x \varphi(\xi)\, d\xi$$

Here $\varphi^*(x)$ represents a sum of integrals analogous to (2.2). The last improper integral in (2.2) can for example be reduced, after the integration, to the form

$$2\int_0^\infty e^{(1+t^2)(z+z_1)} \sum_{m=1}^{\infty} \{(-1)^m \sqrt{1+t^2}\sin[(2m+1)\operatorname{arctg} t] - t\} \times$$

$$\times J_{2m+1}[\rho(1+t^2)] \cos(2m+1)\psi \frac{dt}{1+t^2}$$

This reduces the volume of the computations appreciably, as both types of integrals (for φ and for φ^*) can be computed concurrently.

Computations were performed on a digital computer for an ellipsoid with the aspect ratio of 8 for $v = 0.4\sqrt{2gl}$ and for various depths of immersion $|z_1| > 0.39\nu^{-1}$.

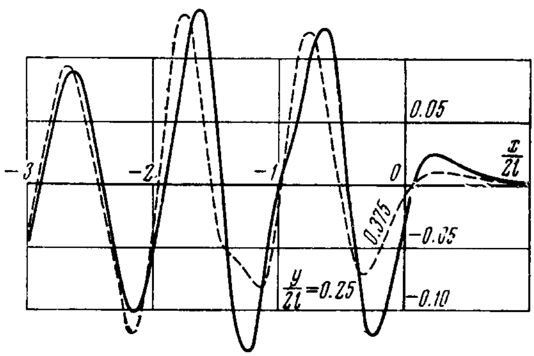

Fig. 1.

For the two points satisfying the condition $x_1 = x_2 + 2l$ we have $\varphi(x_1 - l) = \varphi(x_2 + l)$, therefore for each value of y and z_1 a table of values of φ is held in the computer memory and a relevant selection made from it in order to calculate the right-hand side of the first relation of (3.1). The necessary values of φ^* were obtained in a similar manner and the expression (2.2) was used for $|x| \leqslant 2.5\nu^{-1}$, while (2.3) was employed in the case of large $|x|$.

Figure 1 illustrates, as an example, a section of the free surface for $\frac{1}{2} y / l = 0.25$ and 0.375. Computation of one such curve did not, as a rule, require more than 10

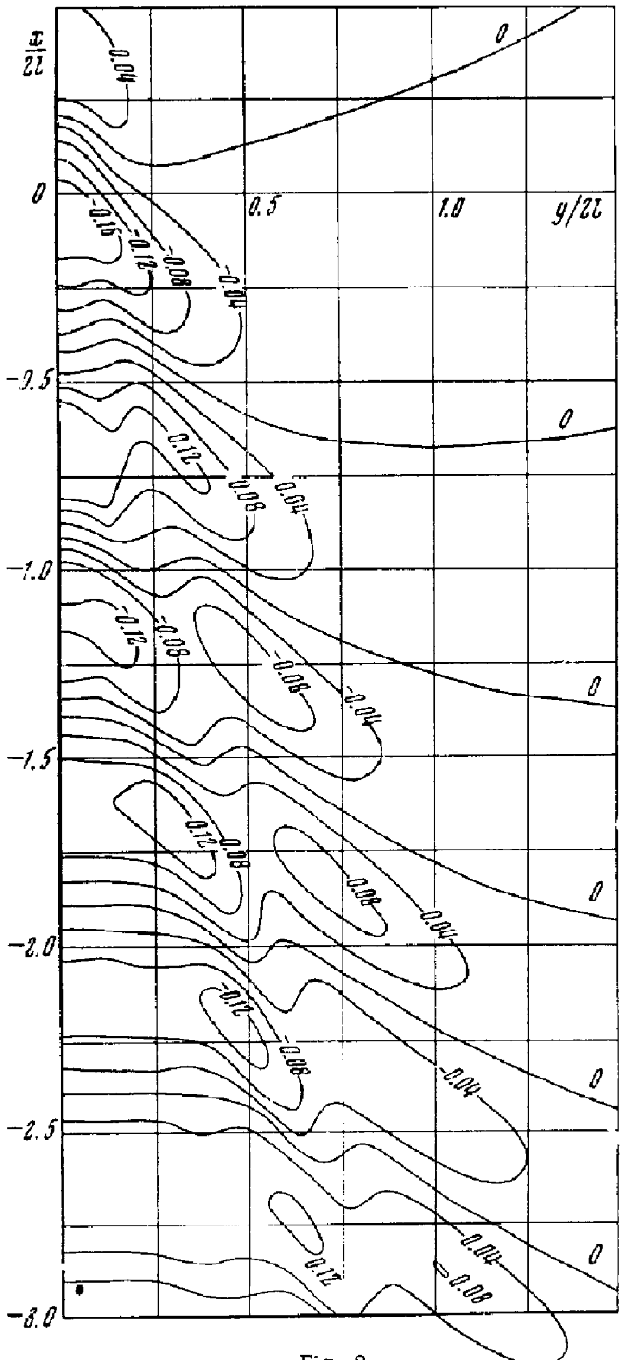

Fig. 2.

minutes. Figure 2 shows a general shape of the free surface obtained by computing the wave profile along nine arbitrary sections. It shows the contour lines spaced at $0.04v^{-1}$ intervals and we see from it that the wave pattern behind a moving ellipsoid is much more complex than one would expect from the asymptotic theory.

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CONVECTIVE INSTABILITY OF A FLUID LAYER IN A MODULATED EXTERNAL FORCE FIELD

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The onset of convection in a layer of an incompressible fluid with free boundaries is considered. The temperature at the layer boundaries, the density of the internal heat sources and the strength of the gravity field are all assumed to be T -periodic. The existence of the critical Rayleigh number and the T-periodicity of the neutral perturbation are proved for the case when the unperturbed temperature gradient is negative throughout the layer. These results are obtained by reducing the linearized problem to an ordinary differential equation in certain Banach space and applying the theory of the linear positive operators [1].

The onset of convection under the action of time-periodic forces is dealt with in [2 - 9]. The stability of equilibrium of a horizontal layer with free and rigid boundaries was investigated and numerical methods were used to determine the